

Homogeneous Equations: —

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where $f(x,y)$ and $\phi(x,y)$ are

homogeneous function of x, y and of the same degree is said to be homogeneous.

Working Rule: — Every homogeneous equation of the above type can be solved by putting $y = vx$ so that

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

After making the substitution for y and $\frac{dy}{dx}$, the given differential equation can be reduced to the form in which the variables are separable. Then after performing the integration we replace y by $\frac{y}{x}$ and the required result follows.

Ex-1. solve $x^2y dx - (x^3 + y^3) dy = 0$

Solution: — The given equation can be written as $(x^3 + y^3) dy = x^2y dx$.

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} \quad \text{put } y = vx \text{ so that}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Hence the equation (1) becomes

$$v + x \frac{dv}{dx} = \frac{x^2 vx}{x^3 + v^3 x^3} = \frac{v}{1 + v^3}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{4v^3} - v = \frac{v \cdot v - v^4}{4v^3} = -\frac{v^4}{4v^3}$$

$$\Rightarrow \frac{dx}{x} = -\frac{4v^3}{v^4} dv$$

Integrating, we get

$$\int \frac{dx}{x} = -\int \frac{4v^3}{v^4} dv$$

$$\Rightarrow \log x = -\int \left(\frac{1}{v^4} + \frac{1}{v} \right) dv$$

$$= -\int (v^{-4} + \frac{1}{v}) dv = \left[\frac{1}{-3v^3} + \log v \right] + C$$

$$= \frac{1}{3v^3} - \log v + C = \frac{x^3}{3y^3} - \log \left(\frac{y}{x} \right) + C$$

$$= \frac{x^3}{3y^3} - \log y + \log x + C$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} + C$$

Which is the required solution.

Ex-2: Solve $x dy - y dx - \sqrt{x^2 + y^2} dx = 0$

Solution: - From the given equation, we have

$$x dy = (y + \sqrt{x^2 + y^2}) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \text{--- (1)}$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$

Hence, the given equation reduces to

$$v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2 x^2}}{x}$$

$$= \frac{Vx + x\sqrt{1+V^2}}{x} = V + \sqrt{1+V^2}$$

$$\Rightarrow x \frac{dv}{dx} = \sqrt{1+V^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+V^2}}$$

Integrating, we have

$$\int \frac{dx}{x} = \int \frac{dv}{\sqrt{1+V^2}}$$

$$\Rightarrow \log x = \log(V + \sqrt{1+V^2}) + \log c$$
$$= \log c(V + \sqrt{1+V^2})$$

$$\Rightarrow x = c(V + \sqrt{1+V^2})$$

$$= c\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) \quad (\text{From } V = y/x)$$

$$= c \frac{y + \sqrt{x^2 + y^2}}{x}$$

$$\therefore x^2 = c(y + \sqrt{x^2 + y^2})$$

Which is the required solution.

Anjani Kumar Singh.